R CA

Club Rates may be had on Application

Mathematics News Letter

SUBSCRIPTION $1.00 PER YEAR Single Copies. . 15c.

Published eight times per year at Baton Rouge, Louisiana.

Dedicated to mathematics in general and to the following aims in particular: (1) a study of the common problems of secondary and collegiate mathematics teaching, (2) a true valuation of the disciplines of mathematics, (3) the publication of high class expository papers on mathematics, (4) the development of greater public interest in mathematics by the publication of authoritative papers treating its cultural, humanistic and historical phases.

VOL. 7 BATON ROUGE, LA., MARCH, 1933 No. 6

CONTENTS

LANGUAGE DISCIPLINE IN MATHEMATICS ?

Orthodox critics may shy at the notion of using mathematics to mould a literary style. But analysis shows that ordinary written language and the language of mathematical symbolism have many basic elements in common. J. J. Sylvester is reported to have said that the first essential in mathematics is language, the second is language, the third is language. One may not be inclined to go quite this far in valuing the importance of the language aspect of mathematics. Yet, a little study of the matter discloses facts seriously suggestive of such a view. Particularly is it true in the analysis and algebra fields—those in which Sylvester did most of his research.

Speaking broadly, the mental steps of the student who uses a Latin dictionary and Latin grammar to prove the identity

Arma virumque cano = I sing of arms and a man

are about the same in kind as those he would take in proving that

$$Log(e^x), \text{ referred to base } e, = x.$$

The test of correctness is basically the same in both cases, namely, that of CONSISTENCY. *In each case is used deduction from definitions, along side of the check against error.*

In similar manner, so long as a translation is to be effected from one written language to another, provided the mind is not too accustomed to the translating process, the intellectual exercises required are but little removed in character from those necessary in much of elementary mathematics.

Again, a mastery of the grammar and idiom of a language has easy correspondence to the memory processes that store for ready use extensive word-formulations of mathematical laws and operations. If it is granted that in either one of these studies is a mental discipline, it must be the same discipline in both.

It is not to be denied that behind and supporting any language of a whole people is a vaster amount of human quality, coloring and history, than is to be found behind the language of mathematics. The cause of this is easily seen. Throughout the history of the world mathematics has been a language understood only by the few. Thus, the humanism behind it, has made relatively small appeal to the great masses of the people. Vergil's line, "I sing of arms and a man," stirs the heroic sense in every heart

of every race. The mathematical statement $x = \log\ (e^x)$, referred to base e, thrills nobody's heart, however great may be the admiration of its logical harmony by the mind that understands it.

BUT THE MIND THAT UNDERSTANDS IT *senses a humanism behind all mathematics that is not the less profound in its appeal because that appeal is to a segment of humanity, rather than to* ALL *humanity. It is the humanism of high intellectual effort of great minds of the race. It is a humanism that is mingled with the very blood of scientists who sacrifice all for* TRUTH—*a humanism of simple hearts and unpretentious but inquiring intellects, without ostentation, the humanism of dynamic, irresistible achievement, of* ETERNAL PROGRESS.—*S.T.S.*

EDITORIAL PERSONNEL OF THE MATHEMATICS NEWS LETTER

Acting upon a conviction that the steadily growing group of News Letter readers, already nation-wide in scope, are entitled to know something about the personnel of our Editorial Board, we have gathered some interesting biographic facts of which the following a: offered as a a second installment.—*S. T. S.*

HERBERT ReBARKER

Graduate, Western Kentucky State Normal School; B. S., M. A., Ph.D., Peabody College; teacher and principal in the elementary and high schools of Kentucky; superintendent public schools, Pontotoc, Mississippi; teacher of mathematics, Peabody Demonstration School; instructor in the teaching of mathematics, Peabody College; director of instruction in mathematics and professor of mathematics, East Carolina Teachers College; author, *A Study of the Simple Integral Processes of Arithmetic.*

H. L. SMITH

Born July 7, 1892, at Pittwood, Illinois. Attended University of Oregon, 1911-13; University of Chicago, 1913-1915; Received degree B. S. at Chicago, 1914; M. S. at Chicago, 1915, and Ph. D. at Chicago, 1926. From 1915 to 1926 taught at Northwestern, Princeton, Cornell, Wisconsin, University of Philippines, Minnesota. During war was balistic computer in Ordnance Department. Has been at Louisiana State University since 1926. Publications in *Transactions of American Mathematical Society, American Journal of Mathematics, Annals of Mathematics, Proceeedings of the National Academy.*

COMPOUND INTEREST*

By RAYMOND GARVER

In a recent issue of the *American Mercury* we read of an old man of seventy, who was found dead of starvation in a shabby rooming house. Investigation showed that he had five thousand dollars in a savings bank which paid three and one-half per cent interest. He had tried to live on his income—a little less than fifteen dollars a month, for he was in terror at the thought of losing his principal. The point of the story was this; had the old man only known, he might, with his five thousand dollars, have purchased a life annuity of fifty three dollars a month. That is, an insurance company would have guaranteed, for five thousand dollars, to pay him fifty-three dollars a month as long as he lived. This sum would have been sufficient for his modest needs, and he might easily have enjoyed ten or fifteen more years of life, free from financial worry.

What does this have to do with our topic this morning? Simply this. If the old man of the story had had any real concept of interest, and what it will do, he need not have starved to death. Of course, he knew that the bank was paying him interest on his deposit, but that was not enough. I am not asking that he know a single formula or equation—I will agree with you that that would be unreasonable—but I am asking whether the knowledge that the income from a sum of money bearing interest can be very materially increased over a considerable period of time by gradually diminishing the principle would not, in this case, have been of vital importance. Knowledge of this sort—that is, the fundamental facts about compound interest—can easily be taught in third-year high school mathematics; while the formulas will, of course, be forgotten unless they are frequently used, the important impressions will ordinarily remain.

It is true that the mathematics involved in life annuities is a little more than mere compound interest. It is equally true that a slight knowledge of the sort which I have just outlined would have led the man to make inquiries, probably at his bank, as to what kind of a plan could be devised to meet his needs. A life annuity would no doubt have been suggested.

Fortunately, with most of us the matter is not one of life and death. But on the other hand it is almost impossible to avoid all contact with

*Prepared for presentation over the radio, March 18, 1933.

interest; one could never have money to invest, nor could he have a life insurance policy of any sort; he could never borrow money, nor buy a house, car or radio on a time payment plan. Shall these contacts with interest be made intelligently, with full knowledge of the underlying principles, or shall they be made haphazardly, depending on the word of an agent, a salesman or a neighbor? The question, I believe, answers itself.

Suppose we now consider a man who has a thousand dollars to invest. He is considering the purchase of a lot, which he can buy for that sum. The real estate agent assures him that in ten years the lot will be worth fifteen hundred dollars. Our prospective purchaser feels that this claim is not unreasonable; perhaps several years ago the lot was worth fifteen hundred dollars, and he is confident that it will again be worth that much in more normal times. Will the lot be a good investment, to buy and hold for ten years, assuming that it can then be sold for fifteen hundred dollars? At first thought, the answer might seem "yes", for purchasing fifteen hundred dollars with a thousand dollars seems attractive enough. A thoughtful investor, however, before giving his answer, will wish to determine how much his thousand dollars will grow to in the same period of ten years if he should simply deposit it in a bank. He will find easily that, if the intersst rate is three and one-half per cent a year, which seems a fair average rate to assume over a period of ten years, his thousand dollars, with no effort at all on his part, will increase to fourteen hundred and ten dollars. Considering the taxes which he would have to pay on the lot, as well as other necessary expenses which would arise from time to time, he will probably decide that the purchase of the lot is not as attractive as it at first appeared. If he believed that he would be able to sell the lot for two thousand dollars, instead of fifteen hundred, at the end of the ten years, the matter would, of course, take on a different aspect. But unfortunately hundreds of people invest in lots, and in many other things—I am using a lot simply as an illustration—, without deciding how long they wish to leave their money invested, without forming any careful estimate as to the probable worth of the property at any future time, or without taking the trouble to calculate the probable return on their investment. Without such steps, investment becomes sheer speculation.

Let us now modify the situation a little. Suppose that our investor buys the lot for a thousand dollars, but, instead of holding it for a rise in price, decides to build on it a house for rent. We will

assume that his total investment in house and lot is six thousand dollars, and that he feels that he should have a seven per cent return on his money. Seven per cent of six thousand dollars is four hundred and twenty dollars, but the owner must pay taxes, certain repair bills, and so on, which will come, we shall say, to one hundred and fifty dollars a year. In all then the owner must take in five hundred and seventy dollars a year, or forty seven dollars and fifty cents a month, to cover expenses and the return on his money. Nothing but simple interest has been used so far, though I should point out that receiving forty-seven dollars and fifty cents a month is not actually the equivalent of five hundred and seventy dollars received at the end of a year, for the monthly receipts can be made to earn interest until the end of the year. A strictly accurate solution of the problem would then be a little more complicated than we have made it; if taxes had to be paid before the end of the year another slight modification would arise. Our figures, however, are essentially correct.

The problem, nevertheless, may not be satisfactorily solved. What will the house and lot be worth after ten years, say? The owner may realize that after that length of time the age of the house will prevent its being rented any longer at the former rate, and he may estimate that his property will then be worth perhaps five thousand dollars, the decrease of a thousand dollars being depreciation in the value of the house. He will now wish to know what rent he must charge if he is to have a seven per cent return on his money *for the next ten years*. It is clear that five hundred and seventy dollars a year is no longer enough, for, while this sum pays his annual expenses and gives him seven per cent of six thousand dollars a year, it will leave him, at the end of ten years, with property worth five thousand dollars, whereas at the beginning of the period he had property worth six thousand dollars. This thousand dollar decrease in his capital must be made up also before he can be said to obtain a seven per cent return on his investment, and it can be made up only by an increase in rental payments. It will not be necessary, however, to raise the rent by one hundred dollars a year; the owner, after taking one hundred and fifty dollars each year for expenses and four hundred and twenty dollars each year for his yield, can deposit the excess at interest in a bank. It is easily found, that if the interest rate is three and a half per cent a year, a deposit of eighty-five dollars a year for ten years will accumulate to the desired one thousand dollars by the end of that period. The rent must then be raised by this amount; that is, our investor

will wish to make sure that he can rent his house for about fifty-five dollars a month, over the next decade, before he carries through the project.

We have, in this example, introduced the important idea of a sinking fund. Whenever a sum of money is to be accumulated at some future time by periodic deposits in a bank, we have the same situation. A man making systematic, periodic deposits to build a fund sufficient to buy his Christmas gifts next year, or to buy a house in ten years, or to educate a child in fifteen years, needs to know the mathematics of sinking funds. But again, a knowledge of the fundamental facts of compound interest is sufficient. An endowment insurance policy is a combination of the sinking fund plan with insurance protection.

Perhaps a brief digression from our discussion of individual problems may not be out of the way here. Every sound bond issue must be protected by a sinking fund, so that when the bonds mature they may be paid. Unfortunately, a perusal of the financial columns leads one to the realization that such protection is not always furnished Of course, in the last two or three years many companies have been honestly unable to meet sinking fund requirements. But in the previous boom years it seems that such requirements were merely overlooked; why bother with a sinking fund when you can always exchange new bonds for the maturing bonds, and indeed sell a few more of the new bonds besides? It will perhaps be some years before such optimistic methods are again in vogue.

Let us turn now to a question that affects thousands of people more closely than anything I have said so far—the question of installment buying. The important thing, of course, is to be able to estimate the rate of interest that you are paying under some installment scheme. Plans can be presented easily which appear reasonable at first sight, but which actually involve an exhorbitant interest rate.

Since there are all sorts of installment buying plans, we must fix some common basis on which to compare them. Essentially, they amount to this; a certain down payment is made, the remaining indebtedness is then paid off in weekly, monthly or quarterly payments, or payments of some sort, regular or irregular. Occasionally you can purchase an article on time payments for the same price you would have to pay if you paid cash. In this case, you are clearly paying no interest at all, but this situation is exceptional. Ordinarily there is a "carrying charge", and when there is, you are paying interest. To

compare different installment plans, it is customary to introduce the idea of effective interest rate. We may describe this something as follows. If a man owes one hundred dollars, and if at the end of a year he pays back one hundred dollars principal and six dollars interest, he is obviously paying interest at the rate of six per cent a year. In other words, he has paid six dollars for the use of one hundred dollars for one year. Now let us say that he is required to pay three dollars interest at the end of six months, and three dollars interest at the end of a year. While he is not paying any more money as interest rate than before, you will, I think, agree that he is paying a higher interest than before, for he does not any longer have the use of one hundred dollars of borrowed money for a year; he has the use of one hundred dollars for the first six months, but only of ninety-seven dollars for the second six months. Just how much higher is the interest rate under this second plan? It is unfortunate that the notion of effective interest rate, which is introduced into financial calculations to answer this question, cannot be defined in what I consider a satisfactory manner in a few words. I shall merely say that if you consider that the borrower pays six dollars interest on an avergae principal of ninety-eight dollars and fifty cents (he has one hundred dollars of borrowed money for six months, and ninety-seven dollars for six months), and work out the interest rate as though this were the actual principal, you will get a little more than six and nine hundredths per cent a year. The formula for effective interest rate, which I am not trying to explain, gives exactly six and nine hundredths per cent. And a person who understands algebra can understand and use this formula, and I think he will feel that it is satisfactory way of comparing the interest burden under different types of installment buying.

This much introduction is necessary for the following statement to have any meaning—under many schemes of installment buying the interest charge paid by the buyer amounts to an effective rate of twenty, or thirty or fifty per cent a year. I have before me an actual plan for buying a used car on time where the rate reaches this last figure. I realize that small debts, and small loans, are expensive to handle, and that a certain brokerage charge is not unreasonable. But when all charges—and they are all interest as far as the debtor is concerned, no matter what the creditor chooses to call them—are sufficient to impose an effective interest burden of fifty per cent a year on the debtor, it is time for the prospective debtor to exercise some care. Certainly a good deal of hardship has been caused by entering thought-

lessly into contracts of this oppressive nature, and certainly such conditions will not be generally improved until borrowers understand the facts well enough to take some effective action.

I have purposely not this morning used a mathematical formula, and I have scarcely used a mathematical term. I hope I have been able to outline a few problems which almost anyone might need to know how to solve. And I hope especially that I have emphasized the fact that the mathematics needed to solve any of these problems will not be beyond a person who has had a good second course in algebra in high school.

NUMERICAL NOTATIONS AND THEIR INFLUENCE ON MATHEMATICS

By D. H. LEHMER
Stanford University

After primitive man had learned to count, his next task was to invent ways of representing and recording whole numbers. This can be done in such a vast number of ways that even today the possibilities have not been exhausted and new ways of expressing numbers are being devised each year. Any good history of mathematics gives in detail the story of man's early attempts at writing numbers. It is not my intention to dwell on this subject from a historical point of view. I merely wish to discuss three types of notation which are in use today.

In any system for writing numbers each number must have one and only one representation. But something more is needed in order that the system have more than historical importance: there must be some operation with numbers to which the system is particularly well adapted. Now the operations most frequently met with are addition and multiplication. Hence the more familiar systems of notation are additive and multiplicative.

The simplest additive notation is the tally. Every number can be written uniquely as a sum of units. Addition is performed by merely placing together the numbers to be added. In this system the fact that six plus seven is thirteen is written

$$111111 + 1111111 = 1111111111111$$

This operation is so simple that it becomes laborious to apply. It is really the basis of all mechanical devices for adding.

The most prevalent additive system today is the decimal notation by which numbers are written to the base ten. In fact we have become such slaves to this method that we have confused the concept of a number itself with what it looks like when written iin the decimal system. We seldom stop to realize that Archimedes, were he alive today, would not understand that $23+59=82$ without some explanation of what these symbols really mean. It never occurs to the numerologist of today that it is possible to destroy utterly some of the mystic properties of numbers by merely writing them to a different base. I recall reading an explanation of the number Π which the numerologist wrote 3.1416. The number 3 stands for the trinity while 1 is the unity of the Trinity; 6 is the perfect number of days in which the earth was created*. If we choose to write Π to the perfect base 6 however we obtain 3.05025. Mathematics itself has not escaped from the influence of the number 10. Many theorems have been announced and problems proposed that depend not on numbers but on the digits of numbers. Dickson devotes a chapter of his History of the Theory of Numbers to results of this nature. The curious example

$$29 \,.\, 83=2407 \qquad 23 \,.\, 89=2047$$

is typical. Such facts and problems are not of paramount importance but belong to the field of mathematical recreations. They have always had a certain fascination however especially among non-professional mathematicians. A prize has been offered recently for the largest list of squares with 9 distinct digits.

The base ten has been forced upon us by physiology. Many practical men have urged the adoption of the base 12 with its many divisors, while mathematicians would prefer a prime number for a base. At any rate it is clear that 10 is by no means the best choice. The smaller the base the smaller are the tables of addition and multiplication which must be committed to memory. On the other hand a smaller base requires more digits to represent a given number. An attempt to compromise these conflicting desiderata leads us to the base 6 or 7 according as we are interested in arithmetic from a practical or a theoretical standpoint. The base two with its simple rules : $1+1=10$, $1\,.\,1=1$ recommends itself in a great many ways. It is a

*At the present time I fail to recollect the true meaning of the number 4. There is also the matter of the decimal point. Perhaps some reader can complete these details.

useful tool in almost every branch of mathematics. However the base ten has become such an integral part of our civilization that any universal change in base is impractical.

An entirely different method of writing numbers, and one which is designed to facilitate the investigation of the multiplicative properties of numbers, arises from the so-called fundamental theorem of arithmetic: Every whole number may be decomposed into the product of powers of prime numbers in one and only one way. Thus we may write

$$65520 = 2^4 \,.\, 3^2 \,.\, 5.7.13 \tag{1}$$

We may even go further and suppress the actual primes leaving them to be implied by a positional notation and record only the exponents of the various primes just as we record only the coefficients of the various powers of ten in writing numbers in the decimal system. Thus

$$65520 = 4,2,1,1,0,1 \tag{2}$$

In this system the nth prime is supposed to be "known" in much the same way that the nth power of ten is "known". The systems illustrated by (1) or (2) must be regarded as more fundamental than the decimal system in which ten is so arbitrary. In either case the primes or their exponents may become very large so we must fall back on the decimal notation again as we did in writing the prime 13 in (1). Of course it is possible in (2) to write the exponents in terms of their prime factors by merely repeating the notation of (2) perhaps several times if necessary. This has never been done however.

Multiplication in the system reduces itself to the addition of the exponents. If instead of adding each pair of exponents we select the larger (or the smaller) we obtain the exponents for the least common multiple (or the greatest common divisor) instead of the product. Thus the product, L.C.M. and G.C.D. of the two numbers 4,2,1,1,0,1 and 2,0,3,5,0,0,0,2 are respectively 6,2,4,6,0,1,0,2; 4,2,3,5,0,1,0,2 and 2,0,1,1. A number of other functions important to the theory of numbers may be easily dealt with in either system (1) or (2) and the majority of theorems presuppose that numbers are expressed in terms of their prime factors. Of course addition in this system is out of the question because we can say little or nothing about the prime factors of a sum. It is easy to derive the left side of (1) from the right. But to go from left to right is notoriously difficult. This one problem occupies a central position in the theory of numbers.

In conclusion I wish to call attention to still another way of expressing numbers which, in a sense, is both additive and multiplicative. This method seems to have originated in China and was considered by one Sun Tsu as early as the first century. The story is told of a wise Chinese general who wished to count the exact number of men in his huge army. He knew that this number was somewhere between 100,000 and 150,000. It would have been too risky and tedious to count the men in the ordinary fashion. So the general commanded his men to arrange themselves in groups of 4. Two men were left over. Groups of 5 were next in order and in this case no men were left over. Grouping by 7, 9, 11, and 13 was tried and in these respective cases 4, 6, 5, and 3 men were left over. With these results the general made a few rapid calculations and announced the number of men to be precisely 145,590. Whether or not we see exactly how this answer was found, it is clear that numbers may be described by the remainders which they leave on division by a selected set of divisors. Furthermore, if these divisors are relatively prime (and it serves no useful purpose to have them otherwise) it is easy to see that this method of description is unique for all numbers less than the product of the divisors. In contradistinction to the prime factor method, it is easier to pass from the decimal system to some Chinese system than to go in the opposite direction although this latter step is by no means formidable.

Those of us who are true worshippers of the decimal system will ask: "Would it not have been better to arrange the army in groups of ten and to observe the number of men left over. This would give the last digit in the answer. By arranging these groups in groups of ten, the next to the last digit would be obtained, and so on until there were no groups left." The answer is no. By using such a practical method the general would not have won for China the distinction of being the source of what was later to become the most effective method for solving certain problems in the theory of numbers. In fact, information about the answers to many interesting problems comes to us in the same way that our general received his. Unfortunately there is often more than one army to count; in some cases thousands or even millions of armies are concerned. These must be arranged according to size. The smallest army is the one we are looking for. I shall not attempt to describe the large class of problems to which this Chinese method is applicable. One of these problems is

that of representing a number as a product of primes. As a recent example of this I may cite the factorization:

$$2^{79}-1=2687\ .\ 202029703\ .\ 1113491139767$$

This result was obtained by sorting armies according to size at the rate of 5,000 armies a second.

It is a familiar fact to the student of algebra or geometry that many a seemingly difficult problem may often become remarkably simple when one makes the right change in variable or the appropriate choice of coordinates. In the same way a suitable system for representing numbers will sometimes facilitate and simplify problems in higher arithmetic. Conversely, he who devises a new numerical notation is sure to discover new properties of numbers and to realize more fully the difference between a symbol for a number and the number itself.

NORMALS TO THE PARABOLA

By JAMES McGIFFERT

A very interesting problem in Analytics arises in connection with the parabola. It is that of determining the number of real normals which can be drawn to a parabola from any point in the plane.

We shall find that three normals may be drawn from points in certain regions, two from points in other regions and only one from many other positions. Let us solve the problem analytically.

In McGiffert's Plane and Solid Analytic Geometry, Articles 54 and 59, we find for the equation of the parabola, with horizontal axis, and opening toward the right, with origin at vertex,

$$y^2=2px \tag{1}$$

and that of the normal,

$$y-y'=-\frac{y'}{p}(x-x') \tag{2}$$

where (x', y') represents the point of contact of the normal with the parabola, and (x,y) represents the current coordinates, that is the coordinates of all points on the normal line, including, of course, the

contact points. Let us choose (m, n) as the point from which we are to draw normal lines to the parabola. We then have, as the condition that (m, n) may be on the normal,

$$n - y' = -\frac{y'}{p}(m - x') \tag{3}$$

Since the point of contact, that is, the point of intersection of the normal with the parabola is (x', y'), we have as the condition that (x', y') may be on the parabola,

$$y'^2 = 2px' \tag{4}$$

Substituting for x', in (3), its value from (4), we have

$$n - y' = -\frac{y'}{p}\left(m - \frac{y'^2}{2p}\right) \tag{5}$$

Clearing of fractions, and rearranging (5), we obtain

$$y'^3 - 2pny' + 2p^2y' - 2p^2n = 0,$$

or

$$y'^3 - 2p(m-p)y' - 2p^2n = 0 \tag{6}$$

Equation (6) is evidently a cubic in y', and gives us the ordinates of the points of contact of the normals with the parabola. We know that this equation has three roots, but Cardan's theory of cubics shows us that the three roots are real and unequal, provided R is negative; real, and at least two equal, if R is zero; and one real, and two imaginary, if R is positive. Now $R = \frac{p^3}{27} + \frac{q^2}{4} = -\frac{\text{discriminant}}{108}$, as determined by the general reduced cubic $y^3 + py + q = 0$.

The R of (6) is, therefore,

$$\frac{4p^4n^2}{4} - \frac{8p^3(m-p)^3}{27}$$

or

$$p^4\left[n^2 - \frac{8(m-p)^3}{27p}\right]$$

We see that the sign of R is the same as that of the quantity in brackets in the preceding expression.

But we know that

$$n^2 - \frac{8(m-p)^3}{27p} = 0$$

is the equation of the Evolute of the parabola, that is, the equation of the centers of curvature of all points of the parabola. But, since p is positive, we see that this expression cannot be negative unless m is greater than p. If m is greater than p, R will be negative only if n^2 is less than $\frac{8(m-p)^3}{27p}$, that is, unless the ordinate of the point (m, n) is numerically less than the ordinate of the point on the evolute, whose abscissa is m.

This means that in order that the three roots of our cubic may be real and unequal, the point from which we draw our normals must be to the right of the evolute, that is, on its convex side.

If R is equal to zero the point from which the normals are drawn is on the evolute, and if R is positive, we see that the point from which the normals are drawn is on the convex side of the evolute. Hence we at once determine whether one, two, or three normals are possible from any given in the plane, by drawing a horizontal line through the point considered, and determining whether this line intersects the evolute to the right of the point, at the point, or to the left of the point given. Algebraically then, we substitute the coordinates of any point in the equation

$$n^2 = \frac{8(m-p)^3}{27p}$$

and if the result is positive, only one real normal can be drawn to the parabola from this point; if the result is zero, two real normals can be drawn, and if the result is negative, three real normals can be drawn, no two being coincident.

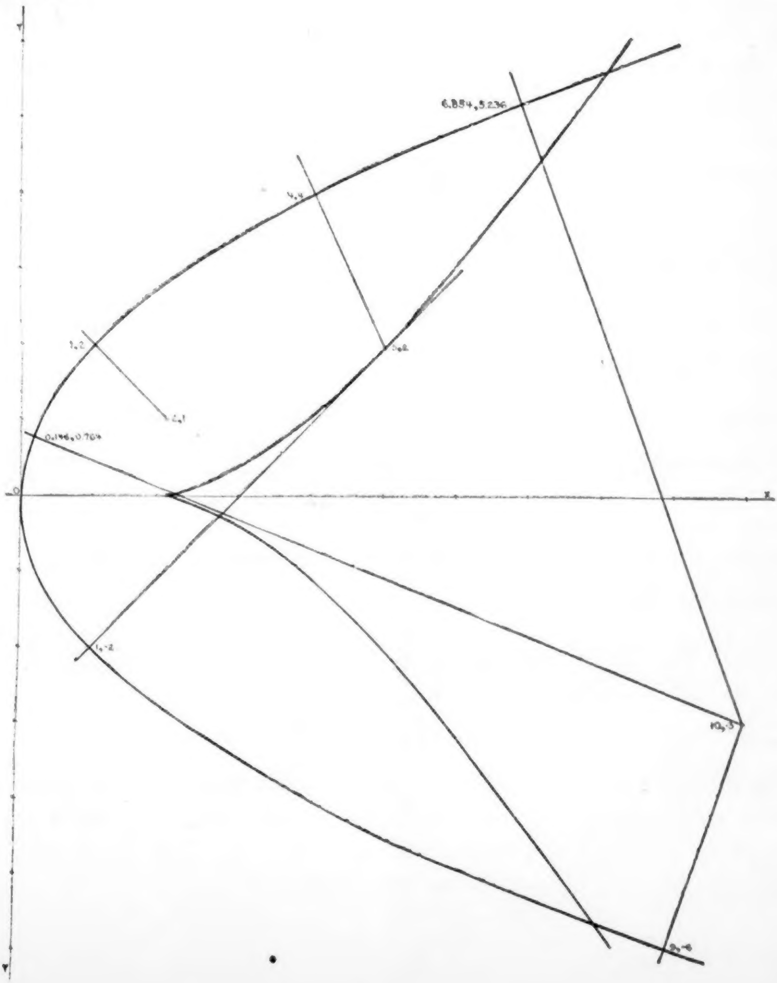
The case of two normals is really the limiting case of three, when the point approaches the evolute, and thus two very nearly coincident normals come to coincide.

We shall illustrate these results by three examples:

I. Let us choose the point (2, 1). and let us use the parabola whose equation is $y^2=4x$, in which $p=2$. Here we have, as the question of the evolute,

$$n^2 - \frac{4(m-2)^3}{27} = 0$$

Substituting in this equation, $m=2$, and $n=1$, we get a positive result, namely, 1. Hence only one normal can be drawn to the parabola, $y^2=4x$, from the point (2, 1). This is shown in the accompanying figure.



II. Next let us choose the point (5, 2).

Substituting 5 for m, and 2 for n, in the equation of the evolute, we get zero, as the result, which shows that the point chosen is on the evolute, and we see from the figure that two normals can be drawn.

III. Finally we choose point (10 − 3)

Substituting these values for m and n in the equation of the evolute, we get a negative result, $-\dfrac{1805}{27}$, and we must then have three normals, each real and different from the others. The figure shows them.

To find the point of contact of each normal, with the parabola, we insert the coordinates of the point from which the normal is to be drawn, in place of m and n in (6), and solve for y′. We then find the corresponding value of x′, from the equation of the parabola.

I. The first point was (2, 1). Putting $m=2$, and $n=1$, and $p=2$, in (6), we get

$$y'^3 - Oy' - 8 = 0,$$

which reduces to $\qquad y'^3 = 8,$

whence $\qquad y' = 2$, and the other two roots are imaginary.

Substituting 2 for y′ in $y'^2 = 4x'$, we get

$$4 = 4x',$$

whence $x' = 1$, and we see that the normal from (2, 1), cuts the parabola at the point (1, 2).

I. For the second point, we make $m=5$, and $n=2$, in (6), and get, as our equation for finding y′,

$$y'^3 - 12y' - 16 = 0,$$

whose roots are 4, −2, and −2. Thus we see that we have apparently three real normals, but two of them coincide, since they both cut the parabola at the point whose ordinate is −2. The three points of intersection then reduce to the two ,(4, 4), (1, −2).

III. For the third point, we substitute 10 for m, and -3 for n, in (6), and get

$$y'^3 - 32y' + 24 = 0,$$

whose roots are -6, $3+\sqrt{5}$, and $3-\sqrt{5}$, or -6, 5.236, and 0.764.

The corresponding values of x are,

$$9,\ \frac{7+3\sqrt{5}}{2},\ \text{and}\ \frac{7-3\sqrt{5}}{2},\ \text{or } 9,\ 6.854,\ \text{and } 0.146.$$

The figure shows the normals from the various points, and displays the parabola and its evolute.

If the paraboloid of revolution be formed, with this parabola for meridian section, we see that three real supports could be built for a paraboloidal reflector of this type, by choosing the point from which they are constructed, to the right of the evolute. This may be of importance in engineering practice.

We chose a specially placed parabola, but this is entirely immaterial.

Whatever may be the position of the curve, we can in an identical manner show that three normals can be drawn from the convex side of the evolute, two, from points on the evolute, and one from points on the concave side.

AN ANALYTIC TREATMENT OF SPHERICAL TRIGONOMETRY

By H. L. SMITH

There are certain obvious advantages to be obtained by combining the courses in trigonometry and analytic geometry. These have led to the recent publication of a textbook along these lines by E. B. Skinner. This book treats only plane trigonometry.

It occurred to the writer that perhaps spherical trigonometry could easily be made a chapter in solid analytic geometry, and it is the primary object of this note to show how this can be done. Incidentally we shall have to clarify some of the basic terms involved.

1. **Definition of Spherical Triangle.** In a book on solid geometry lying before the writer a spherical triangle is defined as

"a portion of a spherical surface bounded by three or more arcs of great circles." Other books consulted say substantially the same. Let us examine this definition. It is of course ambiguous in that it does not state whether the boundary is, or is not, a part of the triangle. But we do not stress this point since there are other and more grave objections. Turning a couple of pages in the book before me I find this theorem: "The sum of any two sides of a spherical triangle is greater than the third." This theorem is untrue if the above is taken as the definition of spherical triangle, as may be seen from the following. Let a be an arc of a circle containing 359°. Let B,C be its end-points and A be one of its poles. Let b,c be quadrants of great circles joining C,B to A respectively. Then the closed curve made up of a,b,c separates the surface of the sphere into two regions either of which, according to the above definition, is a spherical triangle with a,b,c as sides. But it is clear that $b+c<a$, contrary to the above theorem. Nor do the other theorems about spherical triangles "proved" in this book fare any better. Thus the sum of the sides of the above triangle is 539° which is not less than 360°, the side a cannot be supplementary to the opposite angle of the polar triangle if such exists, and so on.

We have seen that the common definition just given will not do. We lay down the following: A spherical triangle is a figure consisting of three arcs of great circles on a sphere, each arc being less than 180° and each arc having its end-points, and no other points, in common with the other two arcs. This is of course what the authors referred to above have in mind, but it is not what they have stated.

Let a,b,c be the great circle arcs of which a given spherical triangle consists, and let the pairs ab,ac,bc have the points C,B,A in common respectively. The points A,B,C, are called *vertices* of the triangle, the sides being a,b,c of course. Let O be the centre of the sphere upon which the given spherical triangle lies. The half-plane with OA as edge which contain B and the half-plane with OA as edge which contains C form a dihedral angle which is called the angle at A(or more briefly the angle A) of the triangle. The angles at B,C are defined similarly.

We note that three distinct points A,B,C of a sphere no two of which are opposite poles and which do not all three lie on a great circle determine a spherical triangle of which they are vertices, and which will be referred to as the spherical triangle ABC.

2. **The Polar Triangle Defined.** Let ABC be a spherical triangle. The great circle of which a is an arc has two poles of which

precisely one lies on the same side of that circle as the point A; let it be called A′. Let B′C′, be similarly defined relative to *b,c,* respectively.

Now no two of the points A′,B′,C′ are opposite poles. For suppose A′,B′, were opposite poles and let *g* be the great circle of which they are poles. From the definition of A′ we have

(1) $A'B = 90°, \quad A'C = 90°$

and from the definition of B′,

(2) $B'A = 90° \quad B'C = 90°$

From (1), B and C lie on *g* and from the first of (2) A also does. But this is not in accordance with the definition of triangle ABC.

It is also true that the points A′,B′,C′ do not all lie on the same great circle. For suppose they did lie on such a great circle, say *g*. Then the relations

$$B'A = 90°, C'A = 90°$$

show that A is one of the poles of *g* since B′,C′ are not 180° apart. Similarly, B,C are poles of *g*. But then at least two of the points A,B,C would coincide, which is not the case.

Finally the points A′,B′,C′ are all distinct, for if A′ were the same as B′ then A,B,C would all lie on the great circle of which $A' = B'$ is the common pole.

It follows from the preceeding that A′,B′,C′ determine uniquely a spherical triangle. It is called the *polar triangle* of ABC.

Books on solid geometry state the following theorem.

If A′ B′ C′ is the polar triangle of ABC, then ABC is the polar triangle of A′ B′ C′.

To prove this it is necessary to show that (a) the point A is at a quadrant's distance from each of the points B′,C′, and (b) the points A,A′ are on the same side of the great circle B′C′. The nearly obvious fact (a) is elaborately proved in these books but the less obvious fact (b) is completely ignored. An analytic proof will be given in the next section.

3. Analytic Representation of a Spherical Triangle and Its Polar Triangle. Let us assume a rectangular coordinate system set up with origin at O the centre of the sphere. Let P,Q,R be points on

half-lines OA,OB,OC respectively and let their coordinates be $(a_1a_2a_3)$, $(b_1b_2b_3)$, $(c_1c_2c_3)$. Then the matrix

$$M = \begin{pmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{pmatrix}$$

represents the triangle ABC in this sense that the elements of each row are proportional to the coordinates of a vertex with positive factor of proportionality. If r is the radius of the sphere, and the elements of any row be each multiplied by r over the square root of the sum of the squares of the elements in that row, then the elements of that row become the coordinates themselves of a vertex.

Let us now build the matrix, called the *adjoint* of M,

$$M' = \begin{pmatrix} A_1 & A_2 & A_3 \\ B_1 & B_2 & B_3 \\ C_1 & C_2 & C_3 \end{pmatrix},$$

in which the element in the i-th row and j-th column is $(-1)^{i+j}$ times the determinant obtained from M by striking out in it the i-th row and j-th column. We wish to show that M′ represents the polar triangle A′ B′ C′ of ABC in the same sense as M represents ABC.

To do this let suppose that the notation is such that the determinant of M is positive. (This can be done since that determinant is not zero since the points PQR are not collinear). The equation of plane OBC is

$$A_1x+A_2y+A_2z=0.$$

Hence if

$$P'=(A_1,A_2,A_3)$$

the line OP′ is $\perp$ to plane OBC. Moreover the points P,P′ are both on the same side of the plane OBC since the expression $A_1\ x+A_2\ y+A_3\ z$ has the same sign $(+)$ for $(xyz)=(A_1A_2A_3)$ as it has for $(xyz)=(a_1a_2a_3)$. Hence the half-line OP′ cuts the sphere in the vertex A′ of the polar triangle A′ B′ C′ and thus $A_1\ A_2\ A_3$ are proportional to the coordinates of A′ with positive factor of proportionality. In similar fashion it can be proved that the second and third rows of M′ represent the vertices B′,C′, respectively. We have thus shown that the adjoint M′ of a matrix M represents the polar triangle of the triangle represented by M.

It follows from a well-known theorem of algebra (see Bocher, Higher Algebra, p. 31, Kowalewski, Die Determinantentheorie p. 80), that the adjoint of M′ is dM, where d is the value of the determinit of M and dM denotes the matrix obtained from M by multiplying each of its elements by d. It is plain that dM represents the same spherical triangle as M, that is, ABC. Hence we have proved that ABC is the polar triangle of its polar A′ B′ C′.

The angle Q′ O R′ measures side a′ of the triangle A′ B′ C′. Since its sides are perpendicular to the half-planes which form the sides of dihedral A of triangle ABC, either a′ and A are equal or supplementary. But since R′ is on the same side of OAC as C and Q′ is on the same side of OAC as B, the side a′ is not equal to A and hence

$$a' + A = 180°.$$

On account of the symmetry of the relation of polarity of triangles, it follows that

$$a + A' = 180°.$$

and we have proved that any side of a spherical triangle is supplementary to the opposite angle of its polar triangle.

4. **The Law of Cosines.** We have

(3) $\cos Q' O R' = (B_1C_1 + B_2C_2 + B_3C_3)/(UV)$ where U,V are positive and such that

$$U^2 = B_1^2 + B_2^2 + B_3^2$$
$$V^2 = C_1^2 + C_2^2 + C_3^2$$

But

(4) $\cos Q' O R' = \cos a' = \cos(180° - A) = -\cos A$

Also $B_1 C_1 + B_2C_2 + B_3C_3$ is the scalar product of the two matrices

$$\begin{pmatrix} c_1 & c_2 & c_3 \\ a_1 & a_2 & a_3 \end{pmatrix}, \quad \begin{pmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{pmatrix}$$

Hence by the Lagrange identity (Kowalewski, p. 72, Satz 26)

$$(5) \quad B_1C_1 + B_2C_2 + B_3C_3 = \begin{vmatrix} a_1c_1 + a_2c_2 + a_3c_3, & b_1c_1 + b_2c_2 + b_3c_3 \\ a_1^2 + a_2^2 + a_3^2 & , a_1b_1 + a_2b_2 + a_3b_3 \end{vmatrix}$$

$$= \begin{vmatrix} \cos b, & \cos a \\ 1, & \cos c \end{vmatrix} = \cos b \cos c - \cos a,$$

if we assume the radius of the sphere to be unity. Also $B_1^2+B_2^2+B_3^2$ is the scalar square of the matrix

$$\begin{pmatrix} c_1 & c_2 & c_3 \\ a_1 & a_2 & a_3 \end{pmatrix}$$

so that as above

$$\text{(5a)}\quad U^2=B_1^2+B_2^2+B_3^2=\begin{vmatrix} c_1^2+c_2^2+c_3^2, & c_2a_2+c_2a_3+c_3a_3 \\ a_1c_1+a_2c_2+a_3dc_3, & a_1^2+a_2^2+a_3^2 \end{vmatrix}$$

$$=\begin{vmatrix} 1, & \cos b \\ \cos b, & 1 \end{vmatrix}=1-\cos^2 b=\sin^2 b,$$

from which

(6) $U=\sin b$

since U is positive and b is less than 180°. Similarly

(7) $V=\sin c.$

On substituting (4), (5), (6), (7) into (3) we get

$$\text{(8)}\qquad \cos A=\frac{\cos a-\cos b\cos c}{\sin b\sin c}$$

or

$$\text{(9)}\qquad \cos a=\cos b\cos c-\sin b\sin c\cos A,$$

which is the law of cosines.

5. **The Law of Sines.** If into the equation

$$\sin A=\surd(1-\cos^2 A)$$

the value of cos A from (8) be substituted, the result simplified, and finally both sides of the equation divided by sin a, we get

$$(\sin A)/(\sin a)=[\surd\,(1-\cos^2 a-\cos^2 b-\cos^2 c+2\cos a\cos b\cos c)]$$
$$/[\sin a\sin b\sin c].$$

The right hand member of this equation is symmetric in a, b, c.

Hence

$$(\sin A)/(\sin a)=(\sin B)/(\sin b)=(\sin C)/(\sin c),$$

which is the law of sines.

6. **Conclusion**. The remaining formulas for the oblique triangle may be obtained from these already found just as in the usual text. The formulas for the right triangle may easily be derived from those for the oblique triangle.

We also note that the reader who does not have access to Kowalewski or cannot read German can verify directly (5), (5a) by actual multiplication or he may consult Dickson, First Course in the Theory of Equations, p. 127, ex. 9.

BOOK REVIEW DEPARTMENT

Edited by
P. K. SMITH

Plane Geometry. By A. M. Welchons and W. R. Krickenberger. Ginn and Co., Chicago, 1933. Price $1.28.

This text is designed to develop the student's ability to follow the reasoning of others, to reason well himself and to appreciate the use of geometric forms in the world about him.

Particular consideration is given to individual differences in students by skillfully arranging the material into minimum, medium, and maximum requisites.

The thirty-eight pages in the introduction are devoted to the very important matter of preparing the student for the formal proof of a theorem by showing him the applications of geometry and acquainting him with the meaning and use of geometric material.

Directions for proving theorems and solving originals are given. The idea of enumerating the possible methods and of selecting the suitable one is stressed with the view of developing a logical method of thinking.

The book is attractively bound. It contains an abundance of good, clear and properly arranged figures. There is no crowding of material on the pages.

Among the other desirable features of this book are: the summary of methods of proof at the end of each chapter, word lists, tests and time schedule for planning work.—*Henry F. Shroeder*.

PROBLEM DEPARTMENT

Edited by
T. A. BICKERSTAFF

This department aims to provide problems of varying degrees of difficulty which will interest anyone who is engaged in the study of mathematics.

All readers, whether subscribers or not, are invited to propose problems and solve problems here proposed.

Problems, and solutions will be credited to their authors.

Send all communications about problems to T. A. Bickerstaff, University, Mississippi.

Problems for Solution

No. 38. Proposed by E. M. Shirley, Louisiana Polytechnic Institute

An arc light a units high stands in the middle of a street and a man b $(b<a)$ units high walks along a sidewalk at c feet per second. State the parametric equations which give the position of the end of the man's shadow at any time t. Along what locus does the shadow move, and at what velocity along this locus? Show analytically that the man's velocity must always be less than that of the end of the shadow. (Suggested by Ex. 33 page 107, Love's *Calculus*.

No. 39. Proposed by P. K. Smith, Louisiana Polytechnic Institute.

A woodsman chops halfway through a tree 2γ feet in diameter. One face of the cut is horizontal and the other is inclined α degrees to the horizontal. Find a formula for the volume of the wood cut out by summing elements with the Planes of their basis perpendicular to the axis of the tree.

No. 40. Proposed by P. K. Smith, Louisiana Polytechnic Institute.

The acceleration at any point of a particle moving with uniform velocity along a curve is directed along the normal at the point.